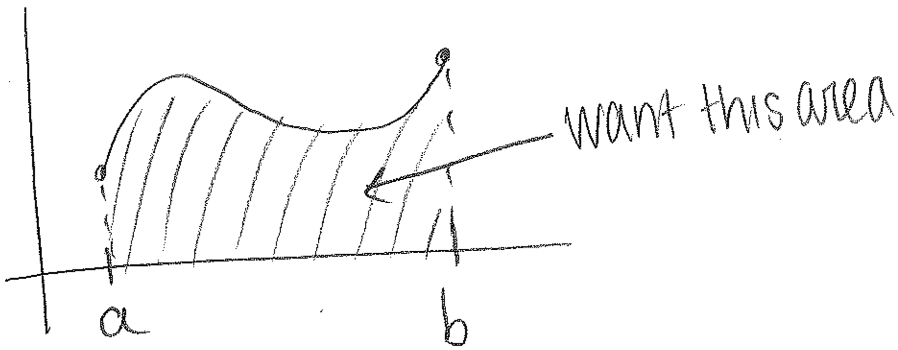
Feb. 21, 2014

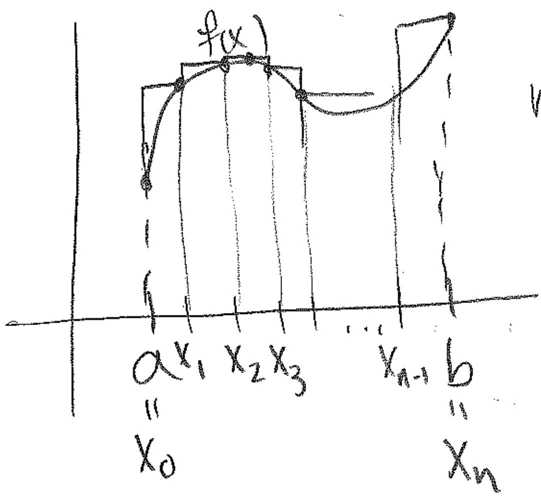
Last time: The Area Problem



So far: estimate area w/ rectangles.

- divide interval $[a,b]$ into n strips

Upper Riemann Sum



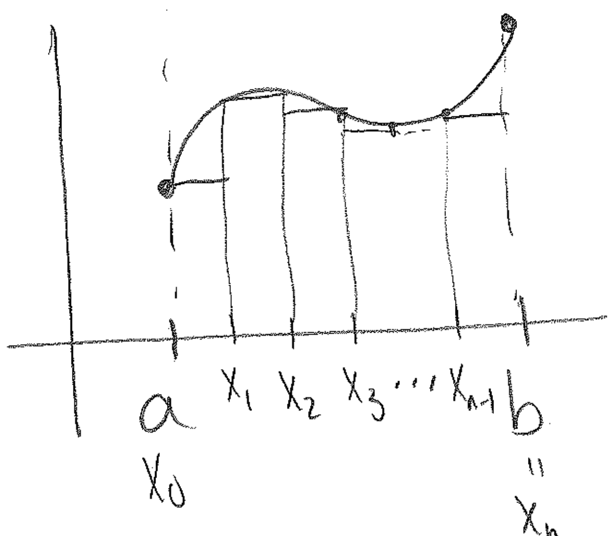
ΔX = width of strips $= \dfrac{b-a}{n}$

* height of rectangle on interval $[x_{i-1}, x_i]$ is max value of $f(x)$ on that interval. (M_i)

Write:

$$\underbrace{U(f,P)}_{\text{upper Riemann Sum}} = M_1\Delta X + M_2\Delta X + \dots + M_n\Delta X$$

Lower Riemann Sum



* height of rectangle on interval $[x_{i-1}, x_i]$ is min value of $f(x)$ on that interval (m_i)

$$L(f,P) = m_1\Delta X + m_2\Delta X + \dots + m_n\Delta X$$

Notation: $\underbrace{U(f, P)}$, $L(f, P)$

f: func

P: how you split up the interval

Sigma Notation

Write sums shorter.

- f a function
- m, n are integers w/ $m \leq n$

$$f(m) + f(m+1) + f(m+2) + \ldots + f(n-1) + f(n)$$

$$= \sum_{i=m}^{n} \underbrace{f(i)}$$

sum $f(i)$ letting $i = m$ then $i = m+1$ and onwards until $i = n$.

Ex: $\sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5$

$$\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2$$

Some facts:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} \frac{1}{a^i}$$

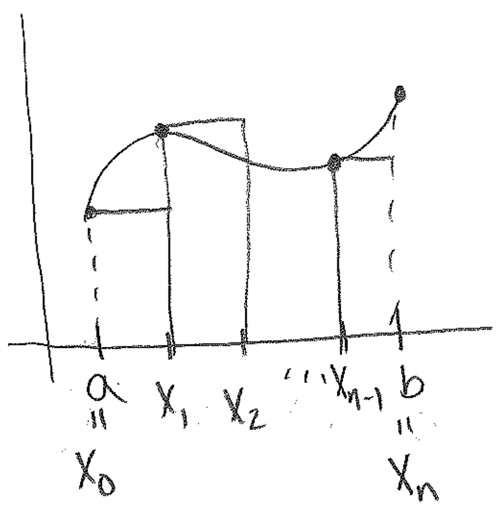
$$\text{So: } U(f,P) = M_1 \Delta x + M_2 \Delta x + \dots + M_n \Delta x = \sum_{i=1}^{n} M_i \Delta x$$

$$L(f,P) = m_1 \Delta x + m_2 \Delta x + \dots + m_n \Delta x = \sum_{i=1}^{n} m_i \Delta x$$

Upper, Lower Riemann Sums are hard to compute in general.

We also have Left and Right Riemann Sums

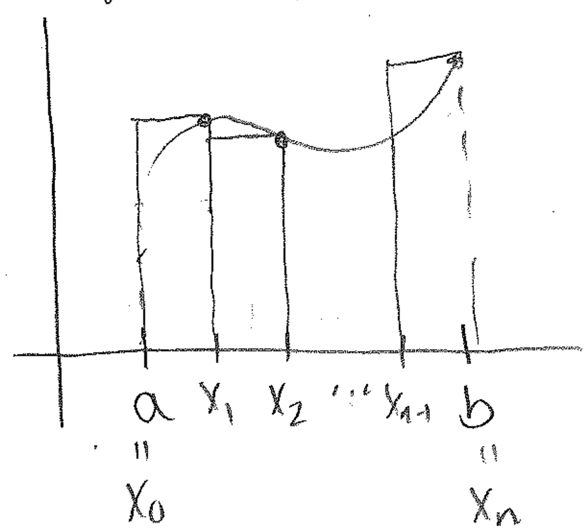
Left Riemann Sum



Height of rectangle is height of function on left side of strip

$$\text{Area} = f(x_0)\Delta x + f(x_1)\Delta x + \dots + f(x_{n-1})\Delta x = \sum_{i=0}^{n-1} f(x_i)\Delta x$$

Right Riemann Sum

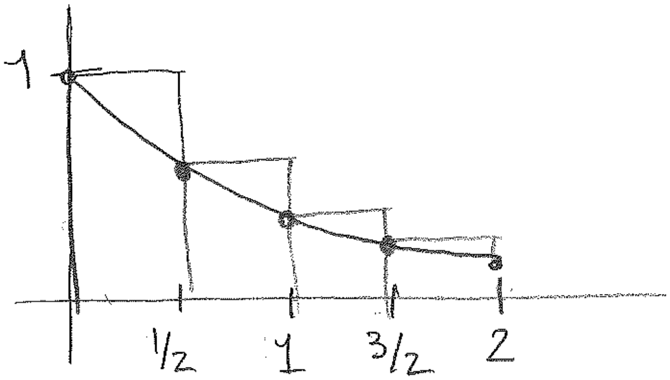


Height of rectangle is height of function on right side of strip

$$\text{Area} = f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x$$

Ex: $f(x) = \frac{1}{3^x}$

area from 0 to 2 using 4 rectangles.



$\Delta x = \frac{b-a}{n}$ bounds, # rectangles

$= \frac{2-0}{4} = 1/2$

In general $x_i = a + i \cdot \Delta x$

$= a + i \cdot \frac{b-a}{n}$

In our problem

$x_i = 0 + i \cdot \frac{1}{2} = \frac{i}{2}$

Left Sum: $\sum_{i=0}^{3} f(x_i)\Delta x$

$= \sum_{i=0}^{3} f\left(\frac{i}{2}\right)\frac{1}{2}$

$= \sum_{i=0}^{3} \frac{1}{3^{(i/2)}} \cdot \frac{1}{2} = \frac{1}{3^0} \cdot \frac{1}{2} + \frac{1}{3^{1/2}} \cdot \frac{1}{2} + \frac{1}{3^1} \cdot \frac{1}{2} + \frac{1}{3^{3/2}} \cdot \frac{1}{2}$

$= 1.0516$

Right Sum: $\sum_{i=1}^{4} f(x_i)\Delta x$

$= \sum_{i=1}^{4} \frac{1}{2 \cdot 3^{(i/2)}} = \frac{1}{2 \cdot 3^{1/2}} + \frac{1}{2 \cdot 3^1} + \frac{1}{2 \cdot 3^{2/3}} + \frac{1}{2 \cdot 3^2} = 0.6071$

Note: For n rectangles: $\Delta x = \frac{b-a}{n} = \frac{2}{n}$, $x_i = 0 + i \cdot \Delta x$

$= \frac{2i}{n}$

Left Sum: $\sum_{i=0}^{n-1} f(x_i)\Delta x$

$= \sum_{i=0}^{n-1} \frac{1}{3^{(2i/n)}} \cdot \frac{2}{n}$

Right Sum: $\sum_{i=0}^{n} \frac{1}{3^{(2i/n)}} \cdot \frac{2}{n}$

$$\boxed{\Delta x = \frac{b-a}{n}, \quad x_i = a + \Delta x \cdot i}$$

Definite Integral

Note: there are some details around whether an integral exists. We will gloss over this.

Notation: $\displaystyle\int_a^b f(x)\,dx =$ the area under $f(x)$ from a to b

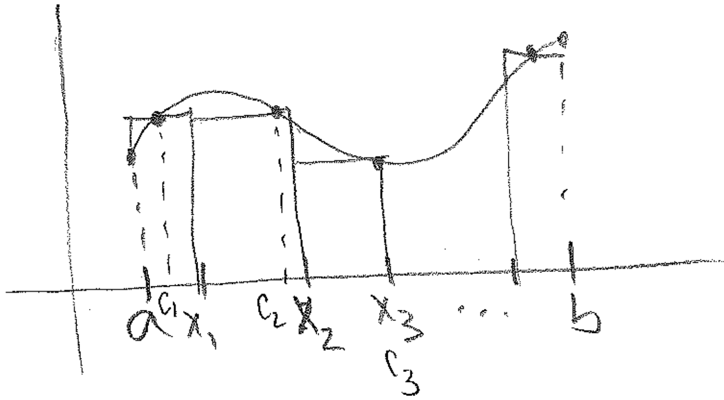
Definition: For f an integrable func on $[a,b]$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i)\,\Delta x\right) \quad \text{(Right Riemann sum)}$$

$$= \underbrace{\lim_{\|P\|\to 0}\left(\sum_{i=1}^{n} f(c_i)\,\Delta x_i\right)}_{\text{general definition the book gives.}}$$

c_i is an pt in $[x_i, x_{i+1}]$

$\|P\|$ = width of fattest rectangle



Idea: Get area by filling it w/ infinitely many rectangles.

Ex: Express $\lim_{n \to \infty} \sum_{i=1}^{n} \underbrace{\frac{3}{n}}_{\text{width}=\Delta x} \sin\left(\underbrace{1+\frac{3i}{n}}_{x_i}\right)$

as a definite integral

bounds on Σ are 1 to n, so right sum.

$\Delta x = \frac{b-a}{n} = \frac{3}{n}$ so $b-a=3$

$x_i = 1 + \frac{3i}{n} = a + \frac{b-a}{n} \cdot i$ $\quad a=1$

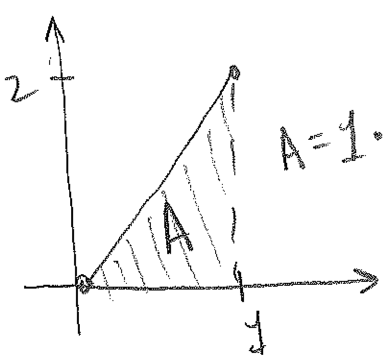
$b-a=3$ so $b=4$

Function is $\sin x$

$$\boxed{\int_1^4 \sin(x)\,dx}$$

Ex: Find $\int_0^1 2x\,dx$ using defin.

$= 1$

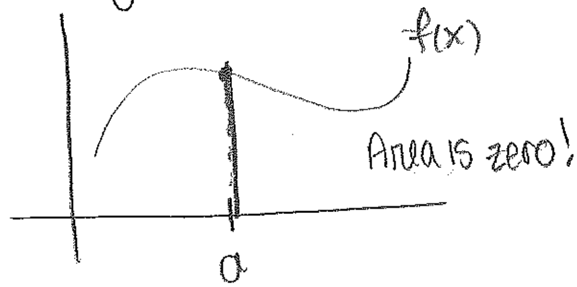


$A = 1 \cdot 2 \cdot \frac{1}{2} = 1$

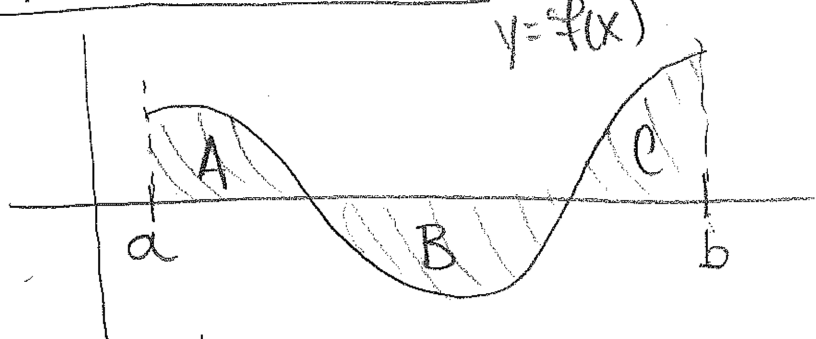
Notation Dissection: $\underbrace{\int_a^b}_{\text{sum from a to b}} \underbrace{f(x)}_{\text{height}}\ \underbrace{dx}_{\text{width}}$

Properties of Definite Integrals

(1) $\int_a^a f(x)\,dx = 0$



IMPORTANT CLARIFICATION:

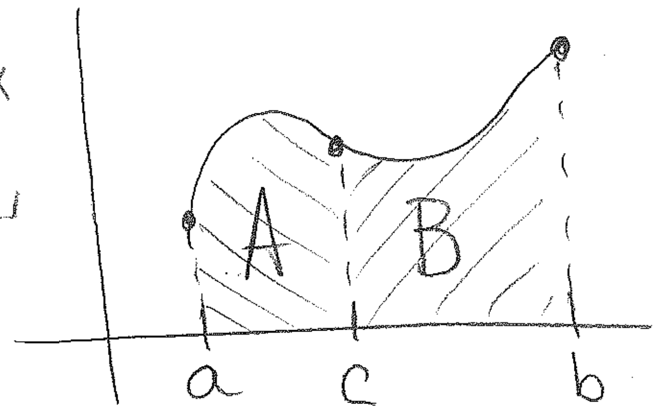


$$\int_a^b f(x)\,dx = A - B + C$$

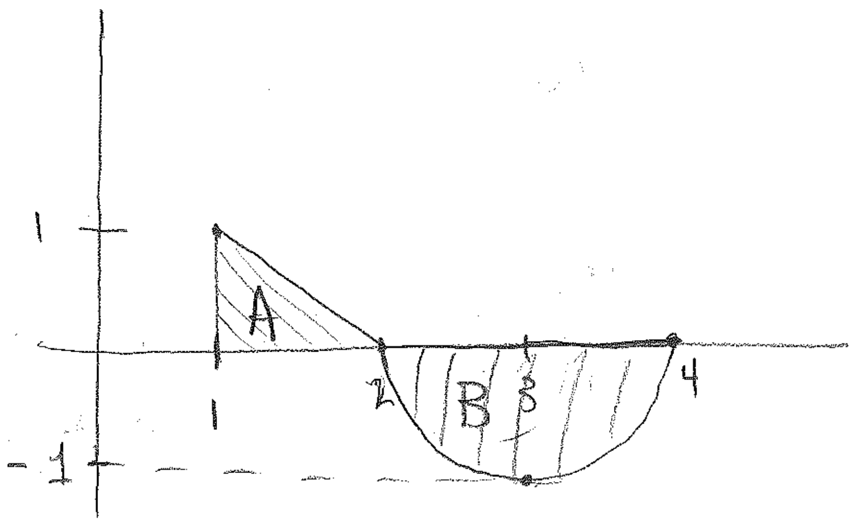
* this is already encoded in all the definitions. You only have worry about it when graphically interpreting area.

(2) $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ (why? $\Delta x = \frac{b-a}{n} = -\frac{a-b}{n}$ comes from def'n)

(3) $\displaystyle\int_a^b f(x)\,dx = \underbrace{\int_a^c f(x)\,dx}_{A} + \underbrace{\int_c^b f(x)\,dx}_{B}$



Partway Example:

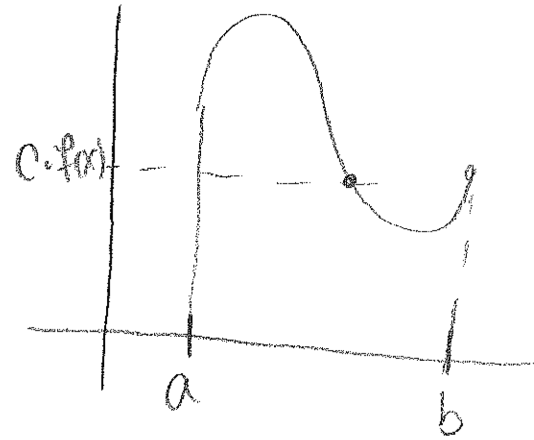


$$\int_1^4 f(x)\,dx = A - B$$

$$= \frac{1}{2}(1)(1) - \frac{1}{2}\pi(1)^2$$

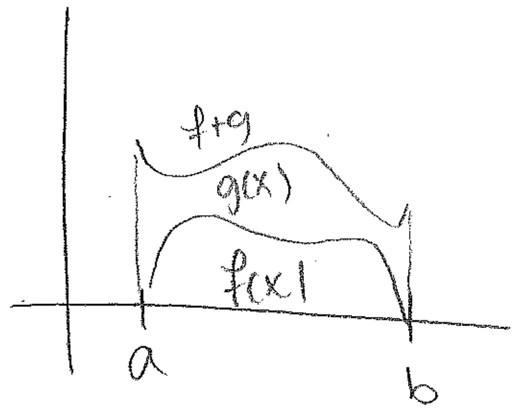
$$\boxed{= \frac{1}{2} - \frac{\pi}{2}}$$

(4) $\int_a^b c\, f(x)\,dx = c\int_a^b f(x)\,dx$



(5) $\int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

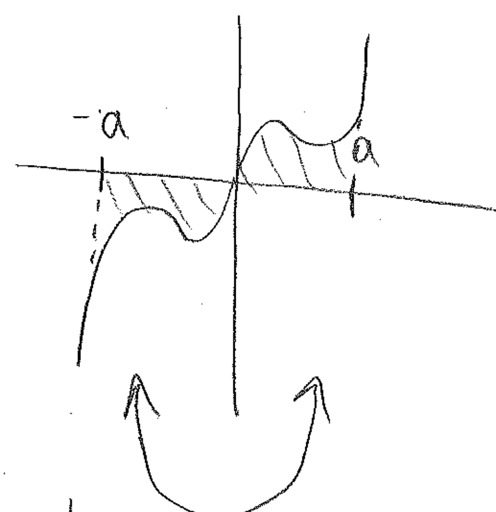
$= \int_a^b f(x)+g(x)\,dx$



(6) (odd functions)

If $f(x)$ is an odd function.

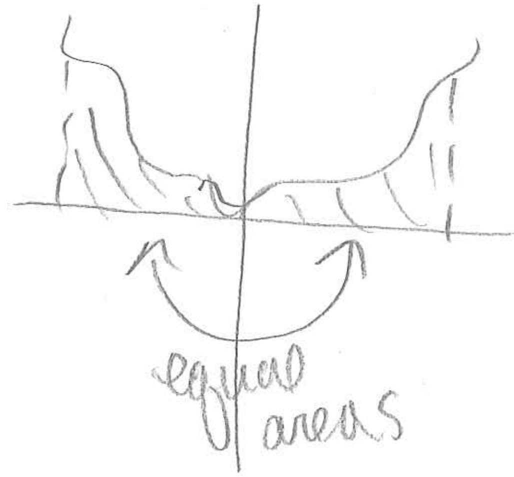
$\int_{-a}^{a} f(x)\,dx = 0$



(7) (even functions)

If f is an even function,

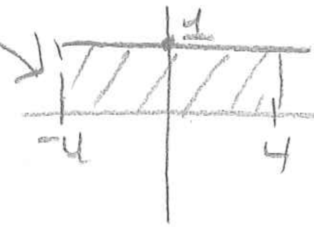
then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$



Ex: $\int_{-4}^{4} (3x^3 - 7x^9 + 7)\,dx$

$= 3\int_{-4}^{4} \underbrace{x^3}_{\substack{\text{odd} \\ 0}} dx - 7\int_{-4}^{4} \underbrace{x^9}_{\substack{\text{odd} \\ 0}} dx + 7\int_{-4}^{4} 1\,dx$

$7 \cdot 1 \cdot 8$

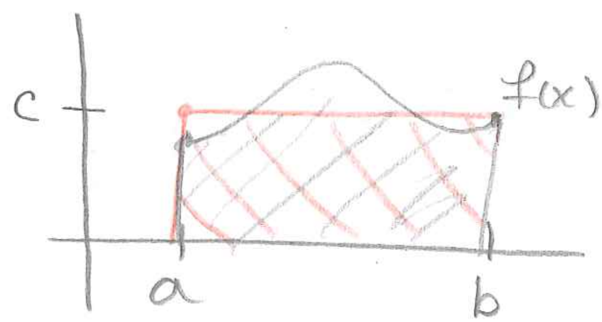


$= 7 \cdot 8 = 56$

Mean Value Thm for Integrals

THM: f cont. on $[a,b]$, there is a # c s.t.

$\int_{a}^{b} f(x)\,dx = \underbrace{c(b-a)}_{\text{area of red rectangle}}$



Avg value of a function on $[a,b]$: $\underbrace{\frac{1}{b-a}}_{\text{width of interval}} \underbrace{\int_{a}^{b} f(x)\,dx}_{\text{all the heights added up.}}$